Proc. Nat. Acad. Sci. USA
Vol. 71, No. 6, pp. 2411–2413, June 1974

A Characterization of Banach Spaces Containing l^1

(pointwise convergence/Boolean independence)

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Communicated by Charles B. Morrey, Jr., March 29, 1974

ABSTRACT **It is proved that a Banach space contains a subspace isomorphic to l^1 if (and only if) it has a bounded sequence with no weak-Cauchy subsequence. The proof yields that a sequence of subsets of a given set has a subsequence that is either convergent or Boolean independent.**

A bounded sequence of elements (f_n) in a Banach space B is said to be *equivalent to the usual l^1-basis* provided there is a $\delta > 0$ so that for all n and choices of scalars $c_1, \ldots, c_n$,

$$\delta \sum_{i=1}^{n} |c_i| \leq \|\Sigma c_i f_i\|. \qquad [1]$$

Of course if (f_n) has this property, then the closed linear span of the f_n's is isomorphic (linearly homeomorphic) to l^1. (f_n) is said to be a *weak-Cauchy sequence* if $\lim_{n\to\infty} b^*(f_n)$ exists for all $b^* \in B^*$, the dual of B.

THE MAIN THEOREM. *Let (f_n) be a bounded sequence in a real Banach space B. Then (f_n) has a subsequence (f'_n) satisfying one of the following two mutually exclusive alternatives:*

(*i*) *(f'_n) is a weak-Cauchy sequence.*

(*ii*) *(f'_n) is equivalent to the usual l^1-basis.*

We note two immediate consequences:

I. *If B is weakly complete (that is, every weak-Cauchy sequence in B converges weakly to an element of B), then B is either reflexive or contains a subspace isomorphic to l^1.*

II. *If B has the Schur property (that is, every weakly convergent sequence in B converges in norm), then every infinite-dimensional subspace of B contains a subspace isomorphic to l^1.*

It is a well-known consequence of the Vitali–Hahn–Saks theorem that $L^1(\mu)$ is weakly complete for any measure μ on a measurable space, while l^1 has the Schur property.

We reformulate the *Main Theorem* as follows:

THEOREM 1. *Let S be a set and (f_n) a uniformly bounded sequence of real-valued functions defined on S. Then (f_n) has a subsequence (f'_n) satisfying one of the following alternatives:*

(*i*) *(f'_n) converges point-wise on S.*

(*ii*) *(f'_n) is equivalent in the supremum norm to the usual l^1-basis.*

The exclusivity of the alternatives of the *Main Theorem* follows from the obvious fact that the usual l^1-basis is not a weak-Cauchy sequence. If (b_n) is a bounded sequence in a Banach space B, we let S denote the unit ball of B^* and then define $f_n(s) = s(b_n)$ for all $s \in S$ and n, to deduce the *Main Theorem* from *Theorem 1*.

We begin the proof of *Theorem 1* with that of the crucial special case of characteristic functions; that is, a sequence (A_n) of subsets of S with $f_n = \chi_{A_n}$ for all n (where $\chi_{A_n}(s) = 1$ if $s \in A_n$; $\chi_{A_n}(s) = 0$ if $s \notin A_n$). (In classical terminology, (A_n) is said to converge if χ_{A_n} converges point-wise.) Our proof of this special case yields that *if (A_n) has no convergent subsequence, then (A_n) has a Boolean independent subsequence* (A_n'); that is, for every pair of nonempty finite disjoint subsets G and B of indices, $\bigcap_{n\in G} A'_n \cap \bigcap_{n\in B} \sim A'_n \neq \phi$. It is easily seen that a Boolean independent sequence (A_n) of subsets of S has the property that (χ_{A_n}) is equivalent to the usual basis of l^1 (see *Proposition 4*).

Because of the technical difficulties encountered in deducing *Theorem 1* from the above special case, we need a generalization of the notion of a convergent sequence of sets. It is also convenient to introduce the following terminology:

By a sequence we shall mean a set of objects indexed by some infinite subset M of the positive integers N; we use the notation $(f_n)_{n\in M}$. *We shall understand by "a subset of M" an infinite subset of M, unless the contrary is explicitly stated.* Given L and M subsets of N, we say that L is almost contained in M if $L \cap \sim M$ is a finite set. Given a sequence $(f_n)_{n\in M}$ and subsets L and Q of M with L almost contained in Q, we call $(f_n)_{n\in L}$ a subsequence of $(f_n)_{n\in Q}$. In the case in which $(f_n)_{n\in M}$ is a sequence of real-valued functions defined on a set S, letting $\{m_1, m_2, \ldots\}$ be a strictly increasing enumeration of M and $s \in S$, we let

$$\overline{\lim_{M}} f_n(s) = \overline{\lim_{j\to\infty}} f_{m_j}(s) \text{ and } \underline{\lim_{M}} f_n(s) = \underline{\lim_{j\to\infty}} f_{m_j}(s).$$

(The point of our terminology, of course, is to avoid explicitly enumerating such sets M whenever it is feasible.)

Definition: Let S be a set, $(A_n, B_n)_{n\in M}$ be a sequence of pairs of subsets of S with $A_n \cap B_n = \phi$ for all n, and X a subset of S. We say that $(A_n, B_n)_{n\in M}$ converges on the set X if every point $x \in X$ either belongs to at most finitely many A_n's, or to at most finitely many B_n's, i.e., either $\lim_{n\to\infty} \chi_{A_n}(x) = 0$ or $\lim_{n\to\infty} \chi_{B_n}(x) = 0$. (When $X = S$, the qualifier "on the set X" may be omitted.) We say that $(A_n, B_n)_{n\in M}$ is independent if for every pair of disjoint finite nonempty subsets G and B of M,

$$\bigcap_{n\in G} A_n \cap \bigcap_{n\in B} B_n \neq \phi. \qquad [2]$$

We note that in the special case where $B_n = S \sim A_n$ for all n and $X = S$, $(A_n, B_n)_{n\in M}$ converges on X if and only if $(A_n)_{n\in M}$ converges. We also note that if $(A_n, B_n)_{n\in M}$ converges on X and $(A_n, B_n)_{n\in L}$ is a subsequence of $(A_n, B_n)_{n\in M}$, then $(A_n, B_n)_{n\in L}$ converges on X; if $(A_n, B_n)_{n\in M}$ converges on each of the sets $X_1, X_2, \ldots$, then $(A_n, B_n)_{n\in M}$ converges on $\bigcup_{i=1}^{\infty} X_i$. Finally, it is an artifact of our definition that every such sequence $(A_n, B_n)_{n\in M}$ converges on the empty set. The special case of *Theorem 1* mentioned above, is an immediate consequence of the next result.

THEOREM 2. *Let $(A_n, B_n)_{n\in N}$ be a sequence of pairs of subsets of a set S with $A_n \cap B_n = \phi$ for all n, and suppose that*

$(A_n, B_n)_{n\in N}$ has no convergent subsequence. Then there is an infinite subset M of N so that $(A_n, B_n)_{n\in M}$ is independent.

Theorem 2, in turn, follows very simply from the next crucial result.

LEMMA 3. *Let $l \geq 1$, $(A_n, B_n)_{n\in N}$ a sequence of pairs of subsets of a set S with $A_n \cap B_n = \phi$ for all n, $X_1, \ldots, X_l$ disjoint subsets of S, and suppose that for each i, $1 \leq i \leq l$, $(A_n, B_n)_{n\in N}$ has no subsequence convergent on X_i. Then there exists a j and an infinite subset M of N so that for each i, $1 \leq i \leq l$, $(A_n, B_n)_{n\in M}$ has no subsequence convergent on $X_i \cap A_j$ and also no subsequence convergent on $X_i \cap B_j$.*

We introduce the following notation: Let $n \in N$ and $\epsilon = \pm 1$; define $\epsilon A_n = A_n$ if $\epsilon = +1$ and $\epsilon A_n = B_n$ if $\epsilon = -1$.

We may deduce *Theorem 2* from *Lemma 3* by the following inductive process: Applying *Lemma 3* for the case $l = 1$, choose n_1 and M_1 a subset of N so that $(A_n, B_n)_{n\in M_1}$ has no subsequence convergent on either A_{n_1} or B_{n_1}. Suppose $n_1 < n_2 < \cdots < n_k$ and M_k have been chosen, so that on each of the 2^k disjoint sets $\bigcap_{j=1}^{k} \epsilon_j A_{n_j}$, $(A_n, B_n)_{n\in M_k}$ has no convergent subsequence, where $\epsilon = (\epsilon_1, \ldots, \epsilon_k)$ ranges over all 2^k choices of signs $\epsilon_i = \pm 1$ all i. Now applying *Lemma 3* for the case $l = 2^k$, choose $n_{k+1} \in M_k$, $n_{k+1} > n_k$, and M_{k+1} a subset of M_k so that for each $\epsilon = (\epsilon_1, \ldots, \epsilon_k)$, $(A_n, B_n)_{n\in M_{k+1}}$ has no subsequence convergent on $\bigcap_{j=1}^{k} \epsilon_j A_{n_j} \cap A_{n_{k+1}}$ and also no subsequence convergent on $\bigcap_{j=1}^{k} \epsilon_j A_{n_j} \cap B_{n_{k+1}}$. This completes the definition of the n_j's and M_j's by induction; it now follows immediately that $M = \{n_1, n_2, \ldots\}$ satisfies the conclusion of *Theorem 2*. We note, incidentally, that the sequence $(A_n, B_n)_{n\in M}$ has the property that it has no subsequence convergent on any nonempty member of the Boolean ring generated by $\{A_n, B_n : n \in M\}$.

We pass now to the proof of *Lemma 3*. The case $l = 1$ is critical, and its proof is constructive in a certain sense. We shall exhibit an algorithm to produce the desired j and M; this algorithm is then used to prove *Lemma 3* by induction. We now suppose $(A_n, B_n)_{n\in N}$ as in the *Definition* and let X be as ubset of S so that $(A_n, B_n)_{n\in N}$ has no subsequence convergent on X. *We shall say that j and M work if $(A_n, B_n)_{n\in M}$ has no subsequence convergent on either $X \cap A_j$ or $X \cap B_j$.* It is obvious that we can assume without loss of generality that $S = X$; we do so.

THE BASIC ALGORITHM. Let n_1 be an arbitrary element of N. If n_1 and N do not work, let N_1 be an arbitrary subset of N with the property that $(A_n, B_n)_{n\in N_1}$ converges on A_{n_1} or B_{n_1}. Suppose $k > 1$ and the subset N_{k-1} of N and the element n_{k-1} of N have been defined. Let n_k be an arbitrary member of N_{k-1} with $n_k > n_{k-1}$. If n_k and N_{k-1} do not work, let N_k be an arbitrary subset of N_{k-1} with the property that $(A_n, B_n)_{n\in N_k}$ converges on A_{n_k} or on B_{n_k}. *We now assert that this process can be continued only a finite number of times. That is, as long as the n_j's and N_j's are selected in the above manner, there must exist a $k \geq 1$ so that n_k works for N_{k-1}* (where $N_0 = N$).

Proof of this assertion: If not, we obtain n_k, N_k, and $\epsilon_{n_k} = \pm 1$, defined for all $k \in N$, with the properties that $n_k > n_{k-1}$, $n_k \in N_{k-1}$, N_k is a subset of N_{k-1}, and $(A_n, B_n)_{n\in N_k}$ converges on $\epsilon_{n_k} A_{n_k}$ (where we put $n_0 = 0$).

Now let $M = \{n_1, n_2, \ldots\}$. Then for every k, $(A_n, B_n)_{n\in M}$ is a subsequence of $(A_n, B_n)_{n\in N_k}$, hence $(A_n, B_n)_{n\in M}$ converges on $\bigcup_{k=1}^{\infty} \epsilon_{n_k} A_{n_k}$. Now we may choose an infinite subset M' of M so that $\epsilon_m = 1$ for all $m \in M'$, or $\epsilon_m = -1$ for all $m \in M'$. Suppose the first possibility. Since $(A_n, B_n)_{n\in M'}$ is a subsequence of $(A_n, B_n)_{n\in M}$ and $\bigcup_{k=1}^{\infty} \epsilon_{n_k} A_{n_k} \supset \bigcup_{n\in M'} \epsilon_n A_n = \bigcup_{n\in M'} A_n$, $(A_n, B_n)_{n\in M'}$ converges on $\bigcup_{n\in M'} A_n$. Since $(A_n, B_n)_{n\in M'}$ does not converge, there must exist an x so that $\{n \in M' : x \in A_n\}$ is infinite and also $\{n \in M' : x \in B_n\}$ is infinite. But then $x \in \bigcup_{n\in M'} A_n$, and hence $(A_n, B_n)_{n\in M'}$ does not converge on $\bigcup_{n\in M'} \epsilon_n A_n$, a contradiction. The proof for the case of the second possibility is the same.

The proof of the assertion of the *Basic Algorithm*, and hence of the case $l = 1$ of *Lemma 3*, is now complete. Suppose *Lemma 3* proved for $l = r$, and let the X_i's and $(A_n, B_n)_{n\in N}$ satisfy its hypotheses for the case $l = r + 1$. Again, we shall say that j and M work if $(A_n, B_n)_{n\in M}$ has no subsequence convergent on either $X_{r+1} \cap A_j$ or $X_{r+1} \cap B_j$. We shall also say that j and M r-work if for every $1 \leq i \leq r$ and $\epsilon = \pm 1$, $(A_n, B_n)_{n\in M}$ has no subsequence convergent on $X_i \cap \epsilon A_j$. By the induction hypothesis, we may choose n_1 and N'_1 a subset of N so that n_1 and N'_1 r-work. If n_1 and N'_1 do not work, choose N_1 a subset of N'_1 so that $(A_n, B_n)_{n\in N_1}$ converges on $A_{n_1} \cap X_{r+1}$ or on $B_{n_1} \cap X_{r+1}$. Suppose $k > 1$, and the subset N_{k-1} of N and the element n_{k-1} of N have been defined. Since $(A_n, B_n)_{n\in N_{k-1}}$ is a subsequence of $(A_n, B_n)_{n\in N}$, we may apply the induction hypothesis to choose an $n_k \in N_{k-1}$ with $n_k > n_{k-1}$ and a subset N'_k of N_{k-1} so that n_k and $(A_n, B_n)_{n\in N'_k}$ r-work. Again, if n_k and $(A_n, B_n)_{n\in N'_k}$ do not work, choose N_k a subset of N'_k so that $(A_n, B_n)_{n\in N_k}$ converges on $A_{n_k} \cap X_{r+1}$ or on $B_{n_k} \cap X_{r+1}$. Now this process cannot be continued indefinitely, since the n_k's and N_k's thus constructed satisfy the criteria of the *Basic Algorithm* and $(A_n, B_n)_{n\in N}$ has no subsequence convergent on X_{r+1}. Thus, there must exist a $k \geq 1$ so that n_k and N'_k work. By construction n_k and N'_k r-work, hence by definition, n_k and N'_k satisfy the conclusion of *Lemma 3*.

This completes the proof of *Lemma 3* and hence of *Theorem 2*. To apply *Theorem 2* to the proof of our *Main Theorem*, we need the following simple sufficient condition for a sequence of functions to be equivalent to the usual l^1-basis.

Proposition 4: Let $(f_n)_{n\in M}$ be a uniformly bounded sequence of real-valued functions defined on a set S and δ and r real numbers with $\delta > 0$. Assume, putting $A_n = \{x: f_n(x) > \delta + r\}$ and $B_n = \{x: f_n(x) < r\}$ for all $n \in M$, that $(A_n, B_n)_{n\in M}$ is independent. Then $(f_n)_{n\in M}$ is equivalent, in the supremum norm, to the usual l^1-basis.

Proof: We shall prove that the "δ" of Eq. **[1]** may be chosen to be $\delta/2$. By multiplying all the f_n's by -1 if necessary, we may assume that $\delta + r > 0$. Let $(c_i)_{i\in M}$ be a sequence of scalars with only finitely many c_i's non-zero and $\Sigma|c_i| = 1$. It suffices to show that there is an s in S with

$$|\Sigma c_i f_i(s)| \geq \delta/2. \qquad [3]$$

Let $G = \{i \in M: c_i > 0\}$ and $B = \{i \in M: c_i < 0\}$. Since Eq. **[2]** holds, we may choose x and y such that $x \in \bigcap_{i\in G} A_i \cap \bigcap_{i\in B} B_i$ and $y \in \bigcap_{i\in B} A_i \cap \bigcap_{i\in G} B_i$. If we suppose first that $r \geq 0$ and set $B' = \{i \in B: f_i(x) > 0\}$, then

$$\sum_{i\in B} c_i f_i(x) \geq \sum_{i\in B'} c_i f_i(x) > -r \sum_{i\in B'} |c_i| \geq \sum_{i\in B} |c_i|(-r). \qquad [4]$$

Similarly,

$$-\sum_{i\in G} c_i f_i(y) \geq \sum_{i\in G} |c_i|(-r). \quad [5]$$

By Eqs. [4], [5], and the definitions of x and y, we thus have

$$\Sigma c_i f_i(x) \geq \sum_{i\in G} |c_i|(\delta + r) + \sum_{i\in B} |c_i|(-r) \quad [6]$$

and

$$-\Sigma c_i f_i(y) \geq \sum_{i\in B} |c_i|(\delta + r) + \sum_{i\in G} |c_i|(-r). \quad [7]$$

It is easily seen that Eqs. [6] and [7] also hold if $r < 0$. Since the sum of the right-hand sides of Eqs. [6] and [7] equals δ, the maximum of the left-hand sides must be at least as large as $\delta/2$. Here we tacitly assumed $G \neq \phi$ and $B \neq \phi$; however, the argument is still valid if we simply replace an intersection over the empty set of indices by S, and a sum over the empty set of indices by 0. Thus, Eq. [3] is established for $s = x$ or $s = y$, so Proposition 4 is proved.

We now assume that S and $(f_n)_{n\in M}$ satisfy the hypotheses of Theorem 1 and fail the first alternative; i.e. $(f_n)_{n\in N}$ has no subsequence point-wise convergent on S. To complete the proof of *Theorem 1*, it suffices to construct $\delta > 0$, a real number r, and a subset M of N so that the hypotheses of the preceding proposition are satisfied. The next two lemmas allow us to find δ and r; their demonstrations involve standard arguments.

LEMMA 5. *For each subset M of N, let*

$$\delta(M) = \sup_{x\in S} (\overline{\lim_{M}} f_m(x) - \underline{\lim_{M}} f_m(x)).$$

Then there exists a subset Q of N so that for all subsets L or Q, $\delta(L) = \delta(Q)$.

Remark: Our standing assumptions imply that $\delta(M) > 0$ for all subsets M of N.

Proof of Lemma 5: For any subsets L and M of N with L almost contained in M, we have that $\delta(L) \leq \delta(M)$. Were the conclusion of *Lemma 5* false, we could choose by transfinite induction, a transfinite family $\{N_\alpha: \alpha < \omega_1\}$ of subsets of N, indexed by the set of ordinals α less than the first uncountable ordinal ω_1, with the property that for all $\alpha < \beta < \omega_1$, N_β is almost contained in N_α and $\delta(N_\beta) < \delta(N_\alpha)$. This is impossible, for as is well known, there does not exist a transfinite strictly decreasing sequence of (positive) real numbers. (To reach a contradiction, simply put $\delta = \inf \{\delta(N_\alpha): \alpha < \omega_1\}$, then choose $(\alpha_n)_{n\in N}$ a sequence of ordinals with $\delta = \lim \delta(N_{\alpha_n})$; for $\beta > \sup_N \alpha_n$, we have $\delta(N_\beta) < \delta$).

Now choose Q a subset of N, satisfying the conclusion of *Lemma 5*, and put $\delta = \delta(Q)/2$; by the above remark, $\delta > 0$.

LEMMA 6. *There exists a subset M' of Q and a rational number r so that for every subset L of M', there is an $x \in S$ satisfying*

$$\overline{\lim_{L}} f_l(x) > \delta + r \textit{ and } \underline{\lim_{L}} f_l(x) < r.$$

Theorem 1 and, hence, our main result follow immediately from *Theorem 2*, *Proposition 4*, and *Lemma 6*. Indeed, let M' satisfy the conclusion of *Lemma 6*, and for each $n \in M'$, let $A_n = \{x \in S: f_n(x) > \delta + r\}$ and $B_n = \{x \in S: f_n(x) < r\}$. The conclusion of *Lemma 6* yields that $(A_n, B_n)_{n\in M'}$ has no convergent subsequence. By *Theorem 2*, we may select a subset M of M' so that $(A_n, B_n)_{n\in M}$ is independent. Then $(f_n)_{n\in M}$ satisfies the hypotheses of *Proposition 4*, hence is equivalent in the supremum norm, to the usual l^1-basis.

Proof of Lemma 6: Suppose not. Let $r_1, r_2, \ldots$ be an enumeration of the rational numbers. Choose L_1 a subset of Q so that

$$\text{for all } x \in S, \overline{\lim_{L}} f_l(x) \leq \delta + r \text{ or } \underline{\lim_{L}} f_l(x) \geq r \quad [8]$$

holds for $L = L_1$ and $r = r_1$. Having chosen the subset L_k of Q, choose $L_{k+1} \subset L_k$ so that Eq. [8] holds for $L = L_{k+1}$ and $r = r_{k+1}$. This defines $L_1 \supset L_2 \supset \ldots \supset L_k, \ldots$ by induction. Now by the standard diagonal procedure, choose an infinite set L with L almost contained in L_k for all $k \in N$. It then follows that Eq. [8] holds for all rational numbers r. Since L is in turn almost contained in Q and Q satisfies the conclusion of *Lemma 5*, $\delta(L) = \delta(Q) = 2\,\delta$. Let $\epsilon = \delta/2$. By the definition of $\delta(L)$, we may choose an $x \in S$ so that

$$\overline{\lim_{L}} f_l(x) - \underline{\lim_{L}} f_l(x) > \delta(L) - \epsilon. \quad [9]$$

Now let $a = \overline{\lim_{L}} f_l(x)$ and $b = \underline{\lim_{L}} f_l(x)$; Eq. [9] may then be expressed in the form

$$a > 2\,\delta - \epsilon + b > b.$$

Choose a rational number r so that $r > b$ and $r - b + \delta < 2\delta - \epsilon = (3/2)\delta$. Thus

$$b < r < r + \delta = (r - b) + \delta + b < 2\delta - \epsilon + b < a.$$

Since we thus have $a > \delta + r$ and $b < r$, Eq. [8] is contradicted.

Q.E.D.

We do not know if the *Main Theorem* holds for complex Banach spaces*. However let B be a complex Banach space, and $(e_n)_{n\in N}$ a sequence in B equivalent to the usual l^1-basis over the real scalars. If there is no subset M of N with $(e_n)_{n\in M}$ equivalent to the usual l^1-basis over the complex scalars, we may choose finite disjoint subsets $B_1, B_2, \ldots$ of N and elements $y_1, y_2, \ldots$ and $z_1, z_2, \ldots$ in B so that $\lim_{j\to\infty} ||y_j - z_j|| = 0$ so that for all j, $||y_j|| = ||z_j|| = 1$ and y_j and iz_j are in the linear span of $\{e_n: n \in B_j\}$. Now for all real scalars a and b and all j, we have that

$$||aiz_j + by_j|| \geq ||(ai + b)z_j|| - ||by_j - bz_j|| = (a^2 + b^2)^{1/2} - |b|\,||y_j - z_j||.$$

It then follows that there is a k so that both sequences $(y_k, iz_k, y_{k+1}, iz_{k+1} \ldots)$ and $(z_k, iz_k, z_{k+1}, iz_{k+1}, \ldots)$ are equivalent to the usual l^1-basis over the real scalars. Consequently $(z_k, z_{k+1}, \ldots)$ is equivalent to the usual l^1-basis over the complex scalars. Hence our *Main Theorem* implies that the result stated in the abstract holds for complex Banach spaces as well.

This research was partially supported by NSF-GP 30798X1.

Addendum. E. Odell and I have shown that a separable Banach space B contains a subspace isomorphic to l^1 provided there exists an element in B^{**} that is not a limit in the B^* topology of a sequence in B. Consequently, B contains an isomorph of l^1 if (and only if) the cardinality of B^{**} is greater than that of the continuum. The proof uses *Proposition 4* and arguments similar to those of *Lemmas 5* and *6*, but does not make use of *Theorem 2* or *Lemma 3*. This result and related ones will appear elsewhere.

* **Note Added in Proof.** This has been resolved in the affirmative by L. Dor.